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On the gravitationally forced motions of a compressible fluid within a horizontally rotating cylinder

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The effect of gravity acting perpendicularly to the stratification induced by rotation in a compressible fluid is investigated. The $\gamma = 1$ approximation used by Gans (1974) in an investigation of the free modes of a rotating gas is used to find an inviscid solution. The boundary layers are calculated, and their suctions are shown to lead to a resonance. The amplitude of the resonance mode is calculated and seen to be of the same order as the simple inviscid solution, so that a corrected solution can be obtained.

The stability of the system is discussed in a simple fashion. It seems likely that a turbulent core will form in a cylinder. The interaction of such a core with the remainder of the fluid is beyond the scope of this paper. Finally a procedure by which a zonal flow field can be found is given.

1. Introduction

The question of the behaviour of rotating compressible fluid systems is relevant to meteorology, astrophysics and engineering. Most attention has been paid to situations where the Boussinesq approximation is valid and there is little information of a global nature available for arbitrary Mach number, though a global approach will eventually be necessary to study the dynamics of such systems with confidence in the results.

In this paper I consider forced motions in a uniformly rotating gas contained in a finite rigid cylinder. The forcing chosen is one realizable in the laboratory: that caused by gravity when the rotation axis of the container is horizontal. This configuration has been chosen in part for simplicity, in part because of its realizability in the laboratory and in part because the free modes of the system are available (Gans 1974, hereafter called I).

The most novel feature of this work is the discovery of a 'resonance' which allows the axially independent forcing function to drive an axially dependent free mode. The amplitude of this driven motion is of the order of the inverse Froude number g/Ω^2L, where g, Ω and L are the acceleration due to gravity, rotation rate and container diameter, respectively. The resonance arises because

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of Ekman suction on the end walls, and this suction is a feature of the compressible case only; *there is no such suction for an unstratified fluid.*

Much of the formulation of this paper depends on I. In § 2 a brief formulation, independent of I, is given, but the reader is referred to I for the details.

There are five relevant parameters involved in this problem. Four of these are inverse Froude number ϵ, an Ekman number E, measuring the influence of viscosity (and more fully discussed below), a Mach number μ, the ratio of the peripheral velocity to the central sound speed, and γ, the ratio of the specific heats. The range of validity of the analysis is $\epsilon \ll E^{\frac{1}{2}} \ll 1, \gamma - 1 \ll 1$. The condition on the Ekman number limits the Mach number, as the Ekman number depends on the Mach number. This point is discussed in § 2 below. The fifth parameter is the length-to-diameter ratio λ.

The plan of the paper is as follows. In § 2 a general formulation, following I, is given. In addition, a forcing term and viscous terms are incorporated. The simplification arising from $\gamma = 1$ as a leading term is demonstrated. The rest of the paper is restricted to the investigation of the $\gamma = 1$ leading term. In § 3 the inviscid forced solution is given, and its boundary layers are calculated. In § 4 the suctions are calculated, and it is shown that the required inviscid corrections lead to formally infinite amplitudes.

In § 5 the correct first-order solution is calculated, giving the amplitude of the resonant mode. The procedure is as follows. The first-order inviscid solution is supposed to be composed of the forced solution calculated in § 3 and an arbitrary amount of the resonant mode uncovered in § 4. The combined boundary layers are found and their suctions are calculated: one of fixed and one of arbitrary amplitude. The correction problem is then formulated, and the condition that the inhomogeneous terms in the correction problem be orthogonal to the resonant mode determines the amplitude of the resonant mode.

In § 6 the stability of the system is discussed briefly, and the analytic results are evaluated for special values of μ and λ.

2. Formulation

The equations of motion will be linearized around a basic state of solid rotation at frequency Ω, and an imposed dimensional temperature distribution

$$T_0'(\varpi) = T_c + \frac{1}{2}\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\frac{\Omega^2\varpi^2}{R}, \tag{2.1}$$

where T_c is the central temperature, γ the ratio of specific heats, R the gas constant, ϖ the radial co-ordinate in a cylindrical (ϖ, ϕ, z) co-ordinate system and α is an introduced parameter. When $\alpha = 1$ the temperature distribution is adiabatic. The prime denotes dimensionality.

The details of the linearization and non-dimensionalization are given in full in I, so I shall merely summarize the results here.

If L is the radius of the container and M the total mass of gas in the container, the density may be scaled by M/L^3, the pressure by $\Omega^2 M/L$, lengths by L, times by Ω^{-1} and velocities by ΩL. λ will be used to denote the length-to-diameter ratio of the container. The dimensionless basic state may then be written as

$$\left.\begin{aligned} T_0(\varpi) &= 1+\frac{1}{2}\frac{\alpha\gamma(\gamma-1)}{1+\alpha(\gamma-1)}\mu^2\varpi^2 = c^2, \\ \mathbf{V}_0(\varpi) &= \varpi\hat{\phi}, \\ \rho_0(\varpi) &\propto \begin{cases} \exp\{\tfrac{1}{2}\mu^2\varpi^2\}, & \alpha = 0 \text{ or } \gamma = 1, \\ T_0^{\,1/\alpha(\gamma-1)} & \text{otherwise.} \end{cases} \end{aligned}\right\} \tag{2.2}$$

The perturbation quantities necessary are the velocity $\mathbf{u}$, pressure p and density ρ. The quantity μ will be referred to as the Mach number, and is defined by

$$\mu^2 = \Omega^2 L^2/\gamma R T_c.$$

The rotation axis is supposed horizontal, so that the gravitational force looks to the rotating fluid like a function of $e^{i\phi}$. The question of the free modes of the system has been extensively examined in I, so that attention here will be restricted to forced velocities, pressures and densities, and these will be supposed to be directly forced, independent of z and t and to vary with ϕ as $e^{i\phi}$.

The equation of state can be used to eliminate ρ in terms of p and $u = \hat{\boldsymbol{\varpi}}.\mathbf{u}$:

$$\rho = (\mu^2/c^2)\,[p - i\rho_0 K\varpi u]. \tag{2.3}$$

Then the conservation of mass and momentum may be written as

$$\left.\begin{aligned} i\rho_0\mathbf{u} + 2\hat{\mathbf{z}}\times\rho_0\mathbf{u} - \hat{\boldsymbol{\varpi}}\varpi\frac{\mu^2}{c^2}p - iK\rho_0\frac{\varpi^2}{c^2}\mu^2 u + \nabla p &= \epsilon\rho_0\hat{\mathbf{x}} + \mathbf{F}_v, \\ i\frac{\mu^2}{c^2}p + \rho_0\frac{\mu^2}{c^2}\varpi u + \rho_0\nabla.\mathbf{u} &= 0, \end{aligned}\right\} \tag{2.4}$$

respectively. In these equations

$$K = \frac{(1-\alpha)(\gamma-1)}{1+\alpha(\gamma-1)}, \quad \epsilon = \frac{g}{\Omega^2 L}, \tag{2.5}$$

and the unit vector $\hat{\mathbf{x}}$ points downwards. $\mathbf{F}_v$ is the viscous force. In I, which was an investigation of the possible free modes of the system, the viscous force was quickly supposed small and discarded. This cannot be done here.

The dimensional viscous force $\mathbf{F}'$ is most easily written in tensor notation as

$$F'_i = \{\eta[(u_{i,j} + u_{j,i}) - \tfrac{2}{3}u_{k,k}\delta_{ij}]\}_{,j}, \tag{2.6}$$

where η is the (dynamic) viscosity, a comma denotes the covariant derivative and δ_{ij} is the Kronecker delta. Using a subscript c to denote quantities at the centre one can write

$$\eta = \eta_c(T/T_c)^{\frac{1}{2}}, \tag{2.7}$$

so that

$$F'_i = \tfrac{1}{2}\eta_c(T/T_c)^{-\frac{1}{2}}\,T_{,j}\{(u_{i,j}+u_{j,i}) - \tfrac{2}{3}u_{k,k}\delta_{ij}\} + \eta[u_{i,jj} + u_{j,ji} - \tfrac{2}{3}u_{k,ki}]. \tag{2.8}$$

Non-dimensionalization introduces the parameter

$$\eta/\rho_0\Omega L^2, \tag{2.9}$$

and then the substitution $p = \rho_0 Q$ and subsequent division by ρ_0 leads to a ϖ-dependent Ekman number

$$E(\varpi) = \eta/\rho_c \rho_0 \Omega L^2, \tag{2.10}$$

which will be supposed to be very much less than unity for all $\varpi\,(0 \leqslant \varpi \leqslant 1)$.

The usual boundary-layer assumption is that derivatives normal to the boundary-layer-dependent variables are much larger than those tangential to the boundaries. Because the fluid under consideration in this paper is compressible an additional assumption, equivalent to boundary-layer thickness small compared with scale height, is necessary. This assumption allows the neglect of the $\nabla\eta$ term, so that the viscous force can finally be approximated by

$$\mathbf{F}_v = E(\varpi)\, \partial^2 \tilde{\mathbf{u}}/\partial n^2, \tag{2.11}$$

where a tilde is used to denote boundary-layer quantities.

The inviscid solution is obtained by reducing the problem to a single second-order ordinary differential equation in $Q = p/\rho_0$, as in I. The boundary-layer corrections are obtained by working directly with the boundary-layer velocities. It is tedious to do this in general and, since the only actual calculations to be performed are those for $\gamma = 1$, the boundary-layer formulation will be deferred until §4 below, and the remainder of this section is given to defining the inviscid problem for forced motion.

The method of solution is similar to that given in I; however, the restriction to directly forced motions makes things easier. In particular the axial velocity component w is identically zero. I set $p = \rho_0 Q e^{i\phi}$ and solve (2.4) for the radial and azimuthal velocity components u and v:

$$\left.\begin{aligned} u &= \frac{i}{3}\frac{1+k_1\varpi^2}{1+k_2\varpi^2}\left\{Q' + \frac{2}{\varpi}Q + K\frac{\mu^2\varpi}{c^2}Q - 3\right\}e^{i\phi}, \\ v &= \frac{1}{3}\frac{1+k_1\varpi^2}{1+k_2\varpi^2}\left\{2Q' + \frac{1}{\varpi}Q + K\frac{\mu^2\varpi}{c^2}Q - 3 + K\frac{\mu^2\varpi^2}{c^2}\right\}e^{i\phi}. \end{aligned}\right\} \tag{2.12}$$

Here a prime denotes differentiation with respect to ϖ, and

$$k_1 = \frac{1}{2}\frac{\alpha\gamma(\gamma-1)}{1+\alpha(\gamma-1)}\mu^2, \quad k_2 = \frac{\gamma-1}{1+\alpha(\gamma-1)}\mu^2\,[\tfrac{1}{2}\alpha\gamma + \tfrac{1}{3}(1-\alpha)]. \tag{2.13}$$

These representations are then substituted into the equation of mass conservation to give a single equation in Q:

$$Q'' + \left[\frac{1}{\varpi} - \frac{\gamma\mu^2\varpi}{1+k_1\varpi^2} - \frac{2k_2\varpi}{1+k_2\varpi^2}\right]Q' - \frac{\mu^2}{\varpi^2}Q - \frac{\gamma\mu^2}{1+k_1\varpi^2}Q = \frac{c_1\varpi}{1-k_1\varpi^2} - \frac{c_2\varpi}{1-k_2\varpi^2}, \tag{2.14}$$

where

$$c_1 = \frac{(1-\alpha)(2+\gamma)+3\alpha\gamma^2}{1+\alpha(\gamma-1)}\mu^2, \quad c_2 = \frac{(\gamma-1)[2/(1-\alpha)+3\alpha\gamma]}{1+\alpha(\gamma-1)}\mu^2.$$

This is to be solved in conjunction with the condition that physical quantities be bounded at the origin and that u, as given above, should vanish on $\varpi = 1$.

The equation can be effectively reduced to first order by the substitution $Q = \varpi\Phi$. After some algebra a first-order equation for Φ' is obtained:

$$\Phi'' + \left[\frac{3}{\varpi} + \frac{\gamma\mu^2\varpi}{1+k_1\varpi^2} - \frac{2k_2\varpi}{1+k_2\varpi^2}\right]\Phi' = \frac{c_1}{1+k_1\varpi^2} - \frac{c_2}{1-k_2\varpi^2}. \tag{2.15}$$

In terms of Φ the boundary condition is

$$\Phi' + 3\frac{1+k_2}{1+k_1}\Phi = 3 \quad \text{on} \quad \varpi = 1. \tag{2.16}$$

It is clear that the problem defined by (2.15) and (2.16), and a boundedness condition at the origin, can now be reduced to quadratures. It should also be clear that such a procedure would produce a solution sufficiently complicated to obscure any information one might hope to obtain. Therefore, in §3, I shall return to the approximation scheme that was so profitable in I: an expansion in powers of $\gamma - 1$, retaining only the zeroth-order term.

3. The forced solution for $\gamma = 1$

The zeroth approximation to Φ satisfies

$$\Phi'' + [3/\varpi + \mu^2\varpi]\,\Phi' = 3, \tag{3.1}$$

subject to the boundary condition

$$\Phi' + 3\Phi = 3 \quad \text{on} \quad \varpi = 1. \tag{3.2}$$

Integrating (3.1) once gives

$$\varpi^3 \exp(\tfrac{1}{2}\mu^2\varpi^2)\,\Phi' = (6/\mu^4)(\tfrac{1}{2}\mu^2\varpi^2 - 1)\exp(\tfrac{1}{2}\mu^2\varpi^2) + A_1, \tag{3.3}$$

and to obtain appropriate behaviour at the origin A_1 must be set equal to $6/\mu^4$. A second integration gives

$$\Phi = \frac{6}{\mu^4}\left\{\frac{1}{2\varpi^2}[1-\exp(-\tfrac{1}{2}\mu^2\varpi^2)] + \tfrac{1}{4}\mu^2[\log\tfrac{1}{2}\mu^2\varpi^2 + E_1(\tfrac{1}{2}\mu^2\varpi^2) - \log\tfrac{1}{2}\mu^2] + A_2\right\}, \tag{3.4}$$

where

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt$$

is the exponential integral and A_2 is to be chosen to satisfy (3.2). The form of the log term has been chosen to take advantage of the identity (Gautschi & Cahill 1964)

$$E_1(z) + \log z = -\gamma_* - \sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n\,n!}, \tag{3.5}$$

which makes it clear that Φ is well behaved at the origin. Here

$$\gamma_* = 0{\cdot}5772156649\ldots$$

is Euler's constant. After solving for A_2 the function Φ may be written as

$$\Phi = \frac{6}{\mu^4}\left\{\frac{1}{2\varpi^2}[1-\exp(-\tfrac{1}{2}\mu^2\varpi^2)] + \frac{\mu^2}{4}[\log\tfrac{1}{2}\mu^2\varpi^2 + E_1(\tfrac{1}{2}\mu^2\varpi^2) - \log\tfrac{1}{2}\mu^2] + \tfrac{1}{6}[\mu^4 + \exp(-\tfrac{1}{2}\mu^2) - 1 - \mu^2 - \tfrac{3}{2}\mu^2 E_1(\tfrac{1}{2}\mu^2)]\right\}. \tag{3.6}$$

The velocity components of the solution are

$$u = -i[f(\varpi)+g(\varpi)]\,e^{i\phi}, \quad v = -[f(\varpi)-g(\varpi)]\,e^{i\phi}, \tag{3.7}$$

where

$$\left.\begin{aligned} f(\varpi) &= \mu^{-4}\varpi^{-2}\,[1-\tfrac{1}{2}\mu^2\varpi^2-\exp(-\tfrac{1}{2}\mu^2\varpi^2)],\\ g(\varpi) &= \frac{3}{2\mu^2}\left[\log\varpi^2+\int_{\frac{1}{2}\mu^2\varpi^2}^{\infty}\frac{e^{-t}}{t}\,dt-\int_{\frac{1}{2}\mu^2}^{\infty}\frac{e^{-t}}{t}\,dt\right]-f(1).\end{aligned}\right\} \tag{3.8}$$

The no-slip boundary conditions are

$$\begin{gathered} u+\tilde{u} = 0 = v+\tilde{v} \quad \text{on} \quad z = \pm\lambda,\\ v+\tilde{v} = 0 \quad \text{on} \quad \varpi = 1, \end{gathered} \tag{3.9}$$

requiring the introduction of boundary-layer velocities. The equations satisfied by these are formed by adding $E\,\partial^2\tilde{\mathbf{u}}/\partial n^2$ to the right-hand side of the momentum equation and noting that $\mathbf{n}.\nabla \gg \mathbf{n}\times\nabla$ for any boundary layer. This assertion allows one to conclude that $\tilde{Q} \ll 1$ and leads to the boundary-layer equations

$$\left.\begin{gathered} i\tilde{u}-2\tilde{v} = E\,\partial^2\tilde{u}/\partial z^2, \quad 2\tilde{u}+i\tilde{v} = E\,\partial^2\tilde{v}/\partial z^2,\\ \partial\tilde{w}/\partial z+\nabla_2.\tilde{\mathbf{u}}+\mu^2\varpi\tilde{u} = 0\end{gathered}\right\} \tag{3.10}$$

near the boundaries $z = \pm\lambda$. Near the boundary $\varpi = 1$ the simpler situation

$$i\tilde{v} = E\,\partial^2\tilde{v}/\partial\varpi^2, \tag{3.11}$$

$$i\tilde{v}/\varpi+\partial\tilde{u}/\partial\varpi = 0 \tag{3.12}$$

occurs. The boundary-layer equations (3.9) and (3.11) are of the same form as their incompressible counterparts (Gans 1970), so that the solutions can be written down directly.

In each case boundary layers of thickness $E^{\frac{1}{2}}$ are appropriate. Near $z = \pm\lambda$

$$\left.\begin{aligned} \tilde{u} &= A_1\exp\left\{\pm\frac{1-i}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right\}+A_2\exp\left\{\pm\frac{3^{\frac{1}{2}}(1+i)}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right\},\\ \tilde{v} &= iA_1\exp\left\{\pm\frac{1-i}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right\}-A_2\exp\left\{\pm\frac{3^{\frac{1}{2}}(1+i)}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right\},\end{aligned}\right\} \tag{3.13}$$

and near $\varpi = 1$

$$\tilde{v} = A\exp\{(1+i)\,(2E)^{-\frac{1}{2}}\,(\varpi-1)\}. \tag{3.14}$$

A_1, A_2 and A are determined by the no-slip boundary conditions:

$$\left.\begin{aligned} A_1 &= ig(\varpi) = (3i/2\mu^2)\,[\log\varpi^2+E_1(\tfrac{1}{2}\mu^2\varpi^2)-E_1(\tfrac{1}{2}\mu^2)]-if(1),\\ A_2 &= if(\varpi) = (i/\mu^4\varpi^2)\,[1-\tfrac{1}{2}\mu^2\varpi^2-\exp(-\tfrac{1}{2}\mu^2\omega^2)],\\ A &= (2/\mu^4)\,[1-\tfrac{1}{2}\mu^2-\exp(-\tfrac{1}{2}\mu^2)].\end{aligned}\right\} \tag{3.15}$$

4. The Ekman suction, its interior correction and resonance

The solutions (3.13)–(3.15), when substituted into (3.10) and (3.12), define 'Ekman suctions': normal boundary-layer velocities smaller by a factor of $E^{\frac{1}{2}}$ than the tangential velocities. This suggests that the entire velocity $\mathbf{v}$ should have been formally expanded in powers of ϵ and $E^{\frac{1}{2}}$:

$$\mathbf{v} = \epsilon\mathbf{u}_1+\epsilon E^{\frac{1}{2}}\mathbf{u}_2+\ldots+\epsilon\tilde{\mathbf{u}}_1+\epsilon E^{\frac{1}{2}}\tilde{\mathbf{u}}_2+\ldots.$$

For these to be the leading two terms, $\epsilon \ll E^{\frac{1}{4}}$. Otherwise, the second term in the expansion would be $O(\epsilon^2)$. As will become clear, this restriction is not necessary to establish the existence and amplitude of the resonance. To compute the amplitude, as will be done in § 5, it is merely necessary that $\epsilon \ll E^{\frac{1}{4}}$.

It should be remarked that the solutions found in § 3 above, and called $\mathbf{u}$ and $\tilde{\mathbf{u}}$, are actually $\mathbf{u}_1$ and $\tilde{\mathbf{u}}_1$. The subscript has been suppressed in the interest of neatness. The subscript 2 will be used when it is required.

The suction terms, part of $\tilde{\mathbf{u}}_2$, can be found by integrating (3.10) and (3.12). After some algebra, the end-wall suction may be written as

$$\tilde{w}_{2s} = \pm \tfrac{1}{2}(\tfrac{1}{6}E)^{\frac{1}{2}}[(3^{\frac{1}{2}}F + \varpi) - i(3^{\frac{1}{2}}F - \varpi)] = \pm W(\varpi), \tag{4.1}$$

where

$$F = 3\varpi[\log \varpi^2 + E_1(\tfrac{1}{2}\mu^2\varpi^2)] + (6/\mu^2\varpi)[1 - \exp(-\tfrac{1}{2}\mu^2\varpi^2)] - \{(2/\mu^2)[1 - \exp(-\tfrac{1}{2}\mu^2)] - 1 + 3E_1(\tfrac{1}{2}\mu^2)]\}\varpi. \tag{4.2}$$

The side-wall suction is more easily calculated:

$$\tilde{u}_{2s} = -(1+i)\mu^{-4}(2E)^{\frac{1}{2}}[1 - \tfrac{1}{2}\mu^2 - \exp(-\tfrac{1}{2}\mu^2)] = U. \tag{4.3}$$

The inviscid correction terms u_2, v_2 and w_2 will satisfy (2.4) with the right-hand side equal to zero, and the boundary conditions

$$u_2(1) + U = 0, \quad w_2(\pm\lambda) \pm W = 0. \tag{4.4}$$

To match W a z-dependent solution will be required. However U can be matched by a z-independent interior solution. This suggests splitting the problem according to the presence or absence of z dependence.

The z-independent solution is both simple and uninteresting. It mimics closely the original solution developed in §§ 2 and 3 and in fact can be incorporated within that solution by adding $Ue^{i\phi}$ to the right-hand side of the first of (2.6). After manipulation the result is a modification of the constraints in (2.15) and (2.16). In the context of this section the formal problem for the z-independent second-order solution $\bar{Q}_2$ would be

$$\left(\frac{\bar{Q}_2}{\varpi}\right)'' + \left[\frac{3}{\varpi} + \mu^2\varpi\right]\left(\frac{\bar{Q}_2}{\varpi}\right)' = \text{constant}, \tag{4.5}$$

with the boundary condition on $\varpi = 1$

$$(\bar{Q}_2/\varpi)' + 3(\bar{Q}_2/\varpi) = \text{constant}, \tag{4.6}$$

the solutions to which have been discussed at length. In what follows attention will be restricted to the z-dependent parts of $\mathbf{u}_2$ and Q_2.

One can eliminate $\mathbf{u}_2$ in terms of Q_2. Seeking solutions for Q_2 proportional to $\exp(i\phi)$ leads to the following boundary-value problem:

$$\left.\begin{aligned} &\frac{\partial^2}{\partial\varpi^2}Q_2 + \left(\frac{1}{\varpi} + \mu^2\varpi\right)\frac{\partial}{\partial\varpi}Q_2 - \frac{1}{\varpi^2}Q_2 - \mu^2 Q_2 - 3\frac{\partial^2}{\partial z^2}Q_2 = 0;\\ &\partial Q_2/\partial\varpi + (2/\varpi)Q_2 = 0 \quad \text{on} \quad \omega = 1,\\ &i\,\partial Q_2/\partial z = \pm W(\varpi) \quad \text{on} \quad z = \pm\lambda. \end{aligned}\right\} \tag{4.7}$$

The latter boundary condition suggests that Q_2 should be an even function of z. To that end Q_2 will be expanded in a series of the form

$$Q_2 = \sum_n A_n Q_n(\varpi) \cos k_n z, \tag{4.8}$$

and the differential equation for each component is

$$Q_n'' + (\varpi^{-1} + \mu^2 \varpi) Q_n' - \varpi^{-2} Q_n - (\mu^2 - 3k_n^2) Q_n = 0; \tag{4.9}$$

k_n is to be chosen such that

$$Q_n' + (2/\varpi) Q_n = 0. \tag{4.10}$$

The remaining boundary condition is

$$\sum_n A_n Q_n(\varpi, k_n) k_n \sin k_n \lambda = -W(\varpi). \tag{4.11}$$

The homogeneous problem for the eigenfunctions $Q_n(\varpi)$ can be rewritten as a Sturm–Liouville problem under the substitutions

$$x = -\tfrac{1}{2}\mu^2\varpi^2, \quad Q_n = \varpi \Phi_n(x), \tag{4.12}$$

viz.

$$[x^2 e^{-x} \Phi_n']' - (k_n^2/2\mu^2) x e^{-x} \Phi_n = 0, \tag{4.13}$$

with

$$-\mu^2 \Phi(-\tfrac{1}{2}\mu^2) + 3\Phi(-\tfrac{1}{2}\mu^2) = 0. \tag{4.14}$$

Thus these eigenfunction–eigenvalue pairs form a complete set and are orthogonal, and the remaining boundary condition can be written formally as

$$\sum_n A_n \Phi(x) k_n \sin k_n \lambda = -\varpi^{-1} W(\varpi) = W(x). \tag{4.15}$$

The functions Φ_n are confluent hypergeometric functions,

$$\Phi(k_n^2/2\mu^2, 2; x) = \Phi(a_n, 2; x). \tag{4.16}$$

A flaw in this formal scheme arises if

$$k_n \lambda = m\pi, \tag{4.17}$$

where n and m are any integers. The k_n are discrete, but tend to infinity with n, so that the equivalent condition, that the length-to-diameter ratio take particular values, viz.

$$\lambda = m\pi/k_n, \tag{4.18}$$

can be approached with arbitrary precision for sufficiently large m and n. There is an infinite set of λ such that $k_n \lambda = n\pi$, and if

$$\int_0^1 \varpi^3 \exp(\tfrac{1}{2}\mu^2\varpi^2) \Phi(a_n, 2; -\tfrac{1}{2}\mu^2\varpi^2) W(-\tfrac{1}{2}\mu^2\varpi^2) d\varpi \neq 0, \tag{4.19}$$

the formal result that $A_n = \infty$ is obtained. In more common terms, the system resonates.

5. Construction of a valid first-order solution

A technique that usually works in cases of resonance is to introduce an arbitrary amount of the resonant solution into the lowest-order result and calculate the magnitude of the resonant solution according to a solvability condition. In this case it is clear that the resonant solution will have to be of the same order as the forced solution, not significantly larger. This contrasts with simpler cases (e.g. Gans 1970), and is true because the upset does not occur at first order.

The resonant mode is characterized by

$$\Phi_n(n^2\pi^2/2\lambda^2\mu^2, 2; -\tfrac{1}{2}\mu^2\varpi^2), \tag{5.1}$$

satisfying (4.13) and (4.14), and the forced solution is given by (3.6). This will be denoted by Φ_F, so that the inviscid solution will be taken to be

$$\Phi = \Phi_F + A\Phi_R = \varpi^{-1}(Q_F + AQ_R), \tag{5.2}$$

where A is to be determined. The subscript R denotes 'resonant'. Q_F and Q_R are to be viewed as functions of ϖ.

The inviscid first-order velocity is now given by (3.7) plus the resonant velocities

$$\left.\begin{aligned} u_R &= -\frac{i}{3}A\left[Q_R' + \frac{2}{\omega}Q_R\right]\cos\frac{n\pi}{\lambda}z\,e^{i\phi}, \\ v_R &= \frac{A}{3}\left[2Q_R' + \frac{1}{\varpi}Q_R\right]\cos\frac{n\pi}{\lambda}z\,e^{i\phi}, \\ w_R &= -\frac{in\pi}{\lambda}AQ_R\sin\frac{n\pi}{\lambda}z\,e^{i\phi}. \end{aligned}\right\} \tag{5.3}$$

A prime denotes differentiation with respect to ϖ.

The introduction of z dependence does not change the boundary-layer structure given in § 3 above. The only qualitative addition is a w_R on $\varpi = 1$. It satisfies the same equation as the azimuthal component. Thus the boundary-layer quantities are the same as those given in (3.13)–(3.15), plus the resonant terms. These are written in the same form as the non-resonant terms.

Near $z = \pm\lambda$

$$\left.\begin{aligned} \tilde{u}_{1R} &= \frac{i}{2}A\left\{\left(Q_R' + \frac{1}{\varpi}Q_R\right)\exp\left[\pm\frac{1-i}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right]\right. \\ &\qquad \left.-\frac{i}{6}\left(Q_R' - \frac{1}{\varpi}Q_R\right)\exp\left[\pm\frac{3^{\frac{1}{4}}(1+i)}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right]\right\}e^{i\phi}, \\ \tilde{v}_{1R} &= -\frac{A}{2}\left\{\left(Q_R' + \frac{1}{\varpi}Q_R\right)\exp\left[\pm\frac{1-i}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right]\right. \\ &\qquad \left.-\frac{1}{6}\left(Q_R' - \frac{1}{\varpi}Q_R\right)\exp\left[\pm\frac{3^{\frac{1}{4}}(1+i)}{(2E)^{\frac{1}{2}}}(z\mp\lambda)\right]\right\}e^{i\phi}. \end{aligned}\right\} \tag{5.4}$$

Near $\varpi = 1$ the result is even simpler:

$$\left.\begin{aligned} \tilde{v}_{1R} &= AQ_R(1)\cos\frac{n\pi}{\lambda}z\,e^{i\phi}\exp\left[\frac{1+i}{(2E)}(\varpi-1)\right], \\ \tilde{w}_{1R} &= \frac{in\pi}{\lambda}AQ_R(1)\sin\frac{n\pi}{\lambda}z\,e^{i\phi}\exp\left[\frac{1+i}{(2E)^{\frac{1}{2}}}(\varpi-1)\right]. \end{aligned}\right\} \tag{5.5}$$

To find the suctions it is then merely necessary to put these results into (3.10) and the appropriate modification of (3.12), viz.

$$\frac{i\tilde{v}_{1R}}{\varpi}+\frac{\partial \tilde{w}_{1R}}{\partial z}+\frac{\partial \tilde{u}_{2R}}{\partial \varpi}=0. \tag{5.6}$$

The end-wall Ekman suction, after some algebra, and use of the differential equation to simplify the ϖ-dependent functions, is

$$\tilde{w}_{2R} = \mp \tfrac{1}{2}(\tfrac{1}{6}E)^{\frac{1}{2}}\,[(3^{\frac{3}{2}}+1)\,k_n^2-2\times 3^{\frac{1}{2}}\mu^2]+i[(1-3^{\frac{3}{2}})\,k_n^2+2\times 3^{\frac{1}{2}}\mu^2] = \mp\, W_R(\varpi). \tag{5.7}$$

A similar, simpler calculation, using the boundary condition on Φ_R, gives

$$\tilde{u}_{2R} = -\tfrac{1}{2}(2E)^{\frac{1}{2}}\,(1+i)\left(1+\frac{n^2\pi^2}{\lambda^2}\right)A\,\Phi_R(1)\cos\frac{n\pi}{\lambda}z. \tag{5.8}$$

The right-hand sides of (5.7) and (5.8) will be denoted by

$$W_R(\varpi) \quad \text{and} \quad U_R\cos\{(n\pi/\lambda)\,z\} \text{ respectively.}$$

It is now possible to return to the analysis following (4.7). The new second-order problem for the correction terms is

$$\frac{\partial^2}{\partial\varpi^2}Q_2+\left(\frac{1}{\varpi}+\mu^2\varpi\right)\frac{\partial}{\partial\varpi}Q_2-\frac{1}{\varpi^2}Q_2-\mu^2Q_2-3\frac{\partial^2}{\partial z^2}Q_2=0, \tag{5.9a}$$

$$\frac{\partial Q_2}{\partial\varpi}+2Q_2=-3iU_R\cos\frac{n\pi}{\lambda}z \quad \text{on} \quad \varpi=1, \tag{5.9b}$$

$$i\frac{\partial Q_2}{\partial z}=\mp\,(W(\varpi)-W_R(\varpi)) \quad \text{on} \quad z=\pm\lambda. \tag{5.9c}$$

The condition that this problem have a solution is that the homogeneous solution

$$Q_R=\varpi\Phi_R\cos\{(n\pi/\lambda)\,z\}\,e^{i\phi}$$

be orthogonal to the inhomogeneous terms. To establish this condition, multiply (5.9*a*) by

$$\varpi\exp(\tfrac{1}{2}\mu^2\varpi^2)\,Q_R(\varpi)\cos\{(n\pi/\lambda)\,z\}\,e^{-i\phi}$$

and integrate over the volume, using the boundary conditions to evaluate the surface integrals. This leads to the solvability condition

$$2\int_0^1 Q_R\exp(\tfrac{1}{2}\mu^2\varpi^2)\,W_R\,\varpi\,d\varpi-\lambda U_R Q_R(1)\exp(\tfrac{1}{2}\mu^2) = 2\int_0^1 Q_R\exp(\tfrac{1}{2}\mu^2\varpi^2)\,W\varpi\,d\varpi. \tag{5.10}$$

This can be solved for A in terms of integrals over $\Phi_R=\varpi Q_R$:

$$\begin{aligned}A = {} & \{[6\times 3^{\frac{1}{2}}\mu^2 I_1+3(3^{\frac{3}{2}}+1-2\times 3^{\frac{1}{2}}\mu^2)\,I_2+\mu^2 I_3] \\ & +i[6\times 3^{\frac{1}{2}}\mu^2 I_1+3(3^{\frac{3}{2}}-1-2\times 3^{\frac{1}{2}}\mu^2)\,I_2-\mu^2 I_3]\} \\ & \times\{3[(3^{\frac{3}{2}}+1)\,k^2-2\times 3^{\frac{1}{2}}\mu^2]\,I_1-3i[(3^{\frac{3}{2}}-1)\,k^2-2\times 3^{\frac{1}{2}}\mu^2]\,I_1 \\ & +3^{\frac{3}{2}}(1+i)\,\lambda(1+n^2\pi^2/\lambda^2)\,\Phi_R^2(1)\}^{-1},\end{aligned} \tag{5.11}$$

where $k = n\pi/\lambda$ and the three integrals are given by

$$\left.\begin{aligned} I_1 &= \int_0^1 \varpi^3 \exp\left(\tfrac{1}{2}\mu^2\varpi^2\right) \Phi_R^2(\varpi)\, d\varpi, \\ I_2 &= \int_0^1 \varpi^3 \exp\left(\tfrac{1}{2}\mu^2\varpi^2\right) \Phi_R(\varpi)\, d\varpi, \\ I_3 &= \int_0^1 \varpi^3 \exp\left(\tfrac{1}{2}\mu^2\varpi^2\right) \Phi_R(\varpi)\, \Phi_R'(\varpi)\, d\varpi. \end{aligned}\right\} \qquad (5.12)$$

From these expressions it is clear that the amplitude decreases at least as fast as n^{-2}, and the graver modes are likely to be more important.

I have been unable to perform the integrations in (5.12) for an arbitrary resonance. However the reader can easily verify that the following 'reasonable' setting of the parameters,

$$\mu = 3^{\frac{1}{2}}, \quad \lambda = \tfrac{1}{2}\pi, \quad n = 1, \quad k = 1, \qquad (5.13)$$

defines a resonant mode for which

$$\Phi(\varpi) = e^{-3/2\varpi^2}. \qquad (5.14)$$

With this simplification

$$I_1 = 0{\cdot}0982610221, \quad I_2 = \tfrac{1}{4}, \quad I_3 = -0{\cdot}1699139284, \quad \Phi_R^2(1) = 0{\cdot}0497870683. \qquad (5.15)$$

Substitution of these into (5.11) gives

$$A = 0{\cdot}2481335818 - 0{\cdot}0940164302i. \qquad (5.16)$$

The accuracy displayed is that of the calculations. It far exceeds the accuracy to be expected in any experiment.

6. Discussion

The existence of a z-dependent response of order ϵ driven by a z-independent forcing of order ϵ has been demonstrated, and the magnitude of the driven mode has been calculated formally. The analysis indicates that the high-order (in radial structure) free modes are likely to be unimportant, but that low-order modes can arise for realizable laboratory situations. Thus the flow is likely to be very different from that predicted by the analysis of §3.

Two further questions present themselves in the context of this work. Is the flow stable? What (axisymmetric) zonal flows can one expect? I shall first address the former question and then give a recipe by which the answer to the latter can be found. Following the recipe is beyond the scope of this paper.

The usual infinitesimal stability analysis is clearly not tractable in the present circumstances. Instead I claim that if the pressure and density gradients are anti-parallel the fluid is *inviscidly* unstable. (Both gradients must be non-zero as well.) Viscous stability is beyond the scope of the paper. With this assertion a sufficient condition for instability is

$$\nabla\rho_T . \nabla p_T < 0. \qquad (6.1)$$

Here ρ_T and p_T are the total density and pressure, respectively. In the present linear analysis (6.1) becomes

$$\nabla p_0 . \nabla \rho_0 + \epsilon \nabla p . \nabla \rho_0 + \epsilon \nabla p_0 . \nabla \rho < 0. \tag{6.2}$$

Equation (2.3), with $K = 0$, leads to

$$\rho = \mu^2 \rho_0 \varpi \Phi e^{i\phi} \tag{6.3}$$

for the solution under consideration. Also

$$\nabla \rho_0 = \mu^2 \varpi \rho_0, \quad \nabla p_0 = \varpi \rho_0, \tag{6.4}$$

so that (6.2) becomes

$$\mu^2 \varpi \rho_0^2 \{\varpi + 2\epsilon[(\mu^2 \varpi^2 + 1)\Phi + \varpi \Phi'] w^{i\phi}\} < 0, \tag{6.5}$$

where it is understood that Φ includes both (3.6) and $A\Phi_R$. The $O(\epsilon)$ part of the expression is a function of ϖ^2, so that, if it does not vanish at $\varpi = 0$, there will be a minimum radius within which (6.5) holds, so that a central core of disturbed fluid is a distinct possibility.

I shall prove below that that part of Φ given by (3.6) has the property that

$$\tfrac{3}{8} \leqslant \Phi_F(0) \leqslant 1, \tag{6.6}$$

increasing monotonically with μ. From the series representation of the confluent hypergeometric function,

$$\Phi_R(0) = 1. \tag{6.7}$$

The actual approximate instability criterion is then

$$\varpi + 2\epsilon\{(\Phi_F(0) + \mathrm{Re}\,(A)\cos\phi - \mathrm{Im}\,(A)\sin\phi\} < 0. \tag{6.8}$$

The maximum value of the bracketed term occurs when

$$\tan\phi = -\mathrm{Im}\,A/(\Phi_F(0) + \mathrm{Re}\,A) \tag{6.9}$$

and is equal to the absolute value of $\Phi_F(0) + A$. For the numerical example given in §5 above $\Phi_F(0) \approx 0{\cdot}484$ and

$$\varpi_{\mathrm{crit}} = 1{\cdot}48\epsilon. \tag{6.10}$$

This result is half as large again as that which would be predicted theoretically ignoring the resonance.

Of interest is the fact that instability does not set in at the 'highest point' $\phi = \pi$. The coefficients of $\cos\phi$ and $\sin\phi$ in (6.8) are positive, so that ϕ must lie in the third quadrant to make the bracket negative. Thus the ambiguity in the arctangent is resolved and the critical angle is seen to be

$$\phi_{\mathrm{crit}} = 187^\circ, \tag{6.11}$$

a shift of 7°.

It is usually true when a rotating fluid is subject to a non-axisymmetric perturbation, as in the case discussed in this paper, that an axisymmetric azimuthal or zonal flow is induced. If the perturbation is $O(\epsilon)$ the azimuthal flow is $O(\epsilon^2)$. No amount of iterating back and forth between boundary layers and interior flows, coupled by suctions, can produce a zonal flow; only the nonlinear interactions can change the ϕ dependence. Unfortunately it is not sufficient to calculate the nonlinear forcing term which is axisymmetric, $\mathbf{u}^*.\nabla\mathbf{u}$, to find the zonal flow. The reader can verify that a second-order zonal flow of the form

$$\bar{\mathbf{v}}_2 = (0, V(\varpi), 0)$$

will satisfy the equations and boundary conditions for *any* $V(\varpi)$. The ambiguity can only be resolved by viscosity, which makes the analysis tedious in the extreme. A brief outline of the procedure follows.

One must find the rectified boundary layers, as well as the (arbitrary) interior flow correct to $O(\epsilon^2)$. This involves both forced and free components, as terms of the form

$$\tilde{\mathbf{u}}_1^*.\nabla\mathbf{u}_1, \quad \tilde{\mathbf{u}}_1^*.\nabla\mathbf{u}_1, \quad \mathbf{u}_1^*.\nabla\tilde{\mathbf{u}}_1$$

must be included in the boundary-layer equations. The results of this calculation will involve the arbitrary function $V(\varpi)$. One then insists that the Ekman suction from the end walls, which is $O(\epsilon^2 E^{\frac{1}{2}})$, must be zero, which allows a solution for $V(\varpi)$.

Note that the relative magnitude of ϵ and $E^{\frac{1}{2}}$ is irrelevant to this process. It is only necessary that both be small. The amplitude of the resonant mode is independent of E and must be included, and the mutual orthogonality of axisymmetric and non-axisymmetric terms assures no interference between the $\epsilon E^{\frac{1}{2}}$ effluxes and the ϵ^2 driving forces.

The remainder of paper is devoted to the proof of the statement regarding the behaviour of $\Phi(0)$ as a function of μ. The reader who finds such things tedious may profitably skip to the references.

$\Phi(0)$ may be written in two forms by using the identity (3.5). In terms of the series

$$\Phi(0) = 1+\frac{1}{\mu^4}[\tfrac{1}{2}-1+\exp(-\tfrac{1}{2}\mu^2)]+\frac{3}{2\mu^2}\sum_{n=1}^{\infty}\frac{(-1)^n(\frac{1}{2}\mu^2)^n}{nn!}. \tag{6.12}$$

It is convenient to change the variable by putting $\zeta = \frac{1}{2}\mu^2$, giving the expression

$$\Phi(0) = \Psi(\zeta) = 1+\frac{1}{4\zeta^2}[\zeta-1+e^{-\zeta}]+\frac{3}{4\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n\zeta^n}{nn!}. \tag{6.13}$$

The reader can easily verify that $\Psi(0) = \frac{3}{8}$.

By using the series representation for the exponential function and separating out the constant terms the function can be rewritten as an alternating series, viz.

$$\Psi(\zeta) = \tfrac{3}{8}+\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n+1)}\frac{\zeta^{n-1}}{n!}. \tag{6.14}$$

This function is monotonic if the derivative is of one sign. Differentiating gives

$$\Psi' = \sum_{n=2}^{\infty}\frac{n-1}{n(n+1)}\frac{(-1)^n\zeta^{n-2}}{n!}, \tag{6.15}$$

which is positive for $\zeta = 0$.

If an alternating series has terms which decrease in magnitude, then the sum lies between two successive partial sums. For this series the ratio between the $(n+1)$th and the nth term is

$$R_n = \frac{n^2}{(n-1)(n+1)(n\pm 2)}\zeta$$

and is less than unity when

$$\zeta < (n-1)(n+1)(n+2)/n^2.$$

(The right-hand side is a monotonically increasing function of n.) The series is clearly positive when the second term is less than the first. This corresponds to $n = 2$ in the formula: $\zeta < 3$. The series is monotonic up to $\zeta = 3$.

It is now easier to switch to the other representation to prove monotonicity for the rest of the range. The other representation is

$$\Psi(\zeta) = 1 + \frac{1}{4\zeta^2}[\zeta - 1 + e^{-\zeta}] - \frac{3}{4\zeta}\left[\gamma_* + \log\zeta + \int_\zeta^\infty \frac{e^{-t}}{t}dt\right]. \tag{6.16}$$

The reader can easily verify that $\Psi(\infty) = 1$.

Differentiating this expression and simplifying gives

$$\Psi' = \frac{1}{4\zeta^2}\left[\frac{2}{\zeta} - 4 + 2e^{-\zeta}\left(1 - \frac{1}{\zeta}\right) + 3\left(\gamma_* + \log\zeta + \int_\zeta^\infty \frac{e^{-t}}{t}dt\right)\right], \tag{6.17}$$

and every term except -4 is positive if $\zeta > 1$. Thus the expression is clearly positive if

$$\zeta > \exp\left[\tfrac{4}{3} - 3\gamma_*\right] = 2{\cdot}129990816 < 3. \tag{6.18}$$

Thus the derivative is positive over the entire range $0 \leqslant \zeta < \infty$, and the statement regarding monotonicity is proved.

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